# Figure 7.6: Mathematics Scenarios to Spark Debate

**Elementary scenario:**

Albert and Bernard have just become friends with Cheryl, and they want to know when her birthday is. Cheryl gives them a list of ten possible dates.

| Month | Date | | | | | |
|---|---|---|---|---|---|---|
| May | | 15 | 16 | | | 19 |
| June | | | | 17 | 18 | |
| July | 14 | | 16 | | | |
| August | 14 | 15 | 17 | | | |

Cheryl then tells Albert and Bernard separately the month (Albert) and the day (Bernard) of her birthday.

**Albert:** I don't know when Cheryl's birthday is, but I know that Bernard doesn't know either.

**Bernard:** At first I didn't know when Cheryl's birthday is, but I know now.

**Albert:** Then I also know when Cheryl's birthday is.

So when is Cheryl's birthday?

**Justify your answer and explain how you arrived there.**

**Intermediate scenario:**

The Paradox of Achilles and the Tortoise is one of a number of theoretical discussions of movement put forward by the Greek philosopher Zeno of Elea in the 5th century BC. It begins with the great hero Achilles challenging a tortoise to a footrace. To keep things fair, he agrees to give the tortoise a head start of, say, 500 meters. When the race begins, Achilles unsurprisingly starts running at a speed much faster than the tortoise, so that by the time he has reached the 500 meter mark, the tortoise has only walked 50 meters further than him. But by the time Achilles has reached the 550 meter mark, the tortoise has walked another 5 meters. And by the time he has reached the 555 meter mark, the tortoise has walked another 0.5 meters, then 0.25 meters, then 0.125 meters, and so on. This process continues again and again over an infinite series of smaller and smaller distances, with the tortoise always moving forward while Achilles always plays catch up.

**How is it possible that Achilles overtakes the tortoise?**

**Secondary scenario:**

The Banach–Tarski paradox is a theorem in a set theoretic geometry that states that a solid ball in three-dimensional space can be split into a finite number of nonoverlapping pieces, which can then be put back together in a different way to yield two identical copies of the original ball.

**Explain how this is possible.**

**Elementary scenario answer:**

To solve this problem, we need to look carefully at the question and then work through each statement to see what we can deduce from it.

- From the question, we know that Cheryl told Albert May, June, July, or August.
- Cheryl told Bernard 14, 15, 16, 17, 18, or 19.

**Albert:** "I don't know when Cheryl's birthday is, but I know that Bernard does not know either."

- Cheryl told Bernard the day of her birthday. There are only two days, 18 and 19, that appear once on Cheryl's list. This means that Cheryl's birthday cannot be May 19 or June 18. If it was, then Bernard would know the answer.
- Remember that Cheryl told Albert the month, and from the statement, we can deduce that he knows that Bernard does not know the birthday. For Albert to be certain that Bernard does not know Cheryl's birthday, the month Cheryl told Albert must not have been May or June.
- Therefore, Cheryl's birthday must be in July or August.

**Bernard:** "At first I didn't know when Cheryl's birthday is, but I know now."

- Bernard has worked out that Cheryl's birthday is in July or August.
- If Bernard now knows Cheryl's birthday, the day Cheryl told him would be the 15th, 16th, or 17th. It cannot be the 14th as this date is a possibility for both July and August, and, as such, Bernard wouldn't know Cheryl's birthday for certain.

**Albert:** "Then I also know when Cheryl's birthday is."

- Albert has worked out that Cheryl's birthday is one of the following:
  - July 16
  - August 15
  - August 17
- If Albert now knows for certain when Cheryl's birthday is, she must have told him the month of July. If it had been August, there would be two possible options.

Therefore, the answer is July 16.

**Intermediate scenario answer:**

Suppose we take Zeno's Paradox at face value for the moment, and agree with him that before I can walk a mile, I must first walk a half mile. And before I can walk the remaining half mile, I must first cover half of it; that is, I must walk a quarter of a mile, and then an eighth of a mile, and then a sixteenth of a mile, and then a thirty-second of a mile, and so on. Well, suppose I could cover all these infinite number of small distances: how far should I have walked? One mile! In other words:

$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \ldots$

At first, this may seem impossible: adding up an infinite number of positive distances should give an infinite distance for the sum, but it doesn't. In this case, it gives a finite sum; indeed, all these distances add up to 1! A little reflection will reveal that this isn't so strange after all: if we can divide a finite distance into an infinite number of small distances, then adding all those distances together should produce the finite distance we started with. (An infinite sum is known in mathematics as an infinite series, and when such a sum adds up to a finite number, we say that the series is summable.)

Now, the resolution to Zeno's Paradox is easy. Think about it this way: Obviously, it will take fixed time to cross half the distance to the other side of a room, say two seconds. How long will it take to cross half of the remaining distance? Half as long—only one second. Covering half of the remaining distance (an eighth of the total) will take only half a second, and so on. So if I cross the room, covering all the infinitely many subdistances and adding up all the time it took to traverse them, it will have taken only four seconds.

Poor old Achilles would have won his race.

**Secondary scenario answer:**

Why is this a paradox? Well, it defies intuition because in our everyday lives we normally never see one object magically turning into two equal copies of itself.

It's because it's not possible in our physical world. The mathematical version of the paradox uses the concept of an immeasurable set. Every object in real life is measurable, because it is the set of a finite number of atoms taking up a finite amount of space. Mathematically, even when finite becomes infinite, you still usually have measurable sets. You really have to try very hard in order to create an immeasurable set.

The Banach-Tarski paradox splits the sphere into a finite number of immeasurable sets of points. The key word is *finite*. In fact, it can be shown that it can be split into just FIVE pieces, one of them being the point at the center. So with the other four pieces, we can separate them into two groups of two, and create an entire sphere out of each group, each the same size as the original sphere.

Though this is impossible to do in real life (because we are bound by atoms), it is possible to make a real-life analogy. This analogy will require basic knowledge of the gas laws, namely, that pressure and volume are inversely related. Here we go:

Consider an easily stretchable balloon with some volume of gas inside it. Now release the gas into a container and divide the gas in the container to fill two balloons. Each new balloon will have one-half the volume of the original. But we're going to introduce a trick. We'll reduce the pressure of the room by half. This causes the balloons to each expand to double its size, so that each is as big as the original. We have reconstructed the paradox!

But wait, you say! Even though each new balloon has the same *volume* as the original, it has only one-half the density. So they're not the same balloon as the original.

That objection is correct for the physical world. But in mathematics, we CAN get two identical spheres out of one. Here's the catch: the mathematical sphere has infinite density. When you cut an infinite density in half, the new density is still infinity. This explains the paradox.